Mesumo Puzzles

Stefan Hollos

In a Mesumo puzzle you have to arrange the digits 1 to 9 in a 3 by 3 grid so that the row, column and diagonal sums match the given values. Each digit must be used once so that the sum of all the digits is $1+2+3+4+5+6+7+8+9 = 45$. This will also equal the sum of the row sums and the sum of the column sums. Here is an example of a completed puzzle.

	7	9	2	18
	6	3	5	14
	4	8	1	13
9	17	20	8	11

If you are familiar with magic squares you will recognize this as being related to the 3 by 3 magic square where all row, column and diagonal sums are equal to 15. All the 3 by 3 magic squares using the digits 1 to 9 are shown below. The 8 squares are related by the symmetries of the grid. Those symmetries are rotations of 0, 90, 180 and 270 degrees about the center as well as reflections about the horizontal, verical and the two diagonals.

	2	7	6	15
	9	5	1	15
	4	3	8	15
15	15	15	15	15

	4	9	2	15
	3	5	7	15
	8	1	6	15
15	15	15	15	15

	8	3	4	15
	1	5	9	15
	6	7	2	15
15	15	15	15	15

	6	1	8	15
	7	5	3	15
	2	9	4	15
15	15	15	15	15

	4	3	8	15
	9	5	1	15
	2	7	6	15
15	15	15	15	15

	6	7	2	15
	1	5	9	15
	8	3	4	15
15	15	15	15	15

	2	9	4	15
	7	5	3	15
	6	1	8	15
15	15	15	15	15

	8	1	6	15
	3	5	7	15
	4	9	2	15
15	15	15	15	15

This shows that in general a set of sums does not uniquely determine the grid arrangement. This is true even when the sums are not all equal. To see this consider the grid:

	k	0	-k	0
	-2k	0	2k	0
	k	0	-k	0
0	0	0	0	0

The row, column and diagonal sums of this matrix are zero so we can add it to a Mesumo grid to change the entries without changing the sums. It's clear then that some of the entries in the grid have to be given in order to make it unique. The question is how many? As shown in the general grid below, the unknowns are $a_1, a_2, \ldots a_9$ and the givens are $s_1, s_2, \ldots s_8$.

	a_1	a_2	a_3	s_1
	a_4	a_5	a_6	s_2
	a_7	a_8	a_9	s_3
s_8	s_4	s_5	s_6	s_7

So we have the following equations that have to be satisfied.

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 45 \tag{1}$$
$$a_1 + a_2 + a_3 = s_1 \tag{2}$$
$$a_4 + a_5 + a_6 = s_2 \tag{3}$$
$$a_7 + a_8 + a_9 = s_3 \tag{4}$$
$$a_1 + a_4 + a_7 = s_4 \tag{5}$$
$$a_2 + a_5 + a_8 = s_5 \tag{6}$$
$$a_3 + a_6 + a_9 = s_6 \tag{7}$$
$$a_1 + a_5 + a_9 = s_7 \tag{8}$$
$$a_3 + a_5 + a_7 = s_8 \tag{9}$$

The matrix for this system of equations is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

The matrix is singular. If you calculate the rank of the matrix you'll find that it has rank=7. This means that a minimum of two of the a_i variables must be given for there to be a unique solution. If we set a_1 and a_6 then the matrix for the remaining system of equations is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

This matrix is invertible so we have the following solution for the remaining unknowns.

$$\begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_7 \\ a_8 \\ a_9 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & -1 & 0 & 1 & -2 & 2 & -1 \\ 0 & 1 & 0 & -1 & 2 & -2 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & -2 & 0 & 2 & -1 & 1 & 1 \\ 0 & 3 & 3 & -3 & 0 & -3 & 0 \\ 0 & -1 & 0 & 1 & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} s_1 - a_1 \\ s_2 - a_6 \\ s_3 \\ s_4 - a_1 \\ s_6 - a_6 \\ s_7 - a_1 \\ s_8 \end{pmatrix}$$

For the case of a magic square all the s_i are equal to 15 so we get the following solution.

$$a_2 = 10 + a_6 - 2a_1 \tag{10}$$
$$a_3 = 5 - a_6 + a_1 \tag{11}$$
$$a_4 = 10 - a_6 \tag{12}$$
$$a_5 = 5 \tag{13}$$
$$a_7 = 5 + a_6 - a_1 \tag{14}$$
$$a_8 = 2a_1 - a_6 \tag{15}$$
$$a_9 = 10 - a_1 \tag{16}$$

Note however that not all solutions produce numbers in the range of 1 to 9. In particular when $a_6 = 2a_1$ we get $a_8 = 0$ and $a_2 = 10$.

We can simplify the solution by setting another number in addition to a_1 and a_6. If we set a_8 then the matrix for the remaining system of equations is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and the solution for the remaining unknowns is

$$\begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_7 \\ a_9 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 1 & 2 & -1 & -2 & -1 \\ 0 & -1 & -2 & 1 & 2 & 1 \\ 0 & 1 & -1 & 2 & 1 & -1 \\ 0 & 2 & 1 & -2 & -1 & 1 \\ 0 & -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 2 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} s_1 - a_1 \\ s_2 - a_6 \\ s_3 - a_8 \\ s_4 - a_1 \\ s_6 - a_6 \\ s_8 \end{pmatrix}$$

The possible row and diagonal sums range from 6 to 24. The way to get those sums are shown in the table below. Note that the number in partheses next to the sum is the number of ways to get the sum.

```
 6 (1): (1,2,3)
 7 (1): (1,2,4)
 8 (2): (1,2,5) (1,3,4)
 9 (3): (1,2,6) (1,3,5) (2,3,4)
10 (4): (1,2,7) (1,3,6) (1,4,5) (2,3,5)
11 (5): (1,2,8) (1,3,7) (1,4,6) (2,3,6) (2,4,5)
12 (7): (1,2,9) (1,3,8) (1,4,7) (1,5,6) (2,3,7) (2,4,6) (3,4,5)
13 (7): (1,3,9) (1,4,8) (1,5,7) (2,3,8) (2,4,7) (2,5,6) (3,4,6)
14 (8): (1,4,9) (1,5,8) (1,6,7) (2,3,9) (2,4,8) (2,5,7) (3,4,7) (3,5,6)
15 (8): (1,5,9) (1,6,8) (2,4,9) (2,5,8) (2,6,7) (3,4,8) (3,5,7) (4,5,6)
16 (8): (1,6,9) (1,7,8) (2,5,9) (2,6,8) (3,4,9) (3,5,8) (3,6,7) (4,5,7)
17 (7): (1,7,9) (2,6,9) (2,7,8) (3,5,9) (3,6,8) (4,5,8) (4,6,7)
18 (7): (1,8,9) (2,7,9) (3,6,9) (3,7,8) (4,5,9) (4,6,8) (5,6,7)
19 (5): (2,8,9) (3,7,9) (4,6,9) (4,7,8) (5,6,8)
20 (4): (3,8,9) (4,7,9) (5,6,9) (5,7,8)
21 (3): (4,8,9) (5,7,9) (6,7,8)
22 (2): (5,8,9) (6,7,9)
23 (1): (6,8,9)
24 (1): (7,8,9)
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The sum of all entries is equal to 45 so the number of possible row sums and column sums is equal to the number of ways to partition 45 into 3 parts using the numbers 6 through 24. There are a total of 50 ways shown in the table below.

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 (6,15,24)  (6,16,23)  (6,17,22)  (6,18,21)  (6,19,20)
 (7,14,24)  (7,15,23)  (7,16,22)  (7,17,21)  (7,18,20)  (7,19,19)
 (8,13,24)  (8,14,23)  (8,15,22)  (8,16,21)  (8,17,20)  (8,18,19)
 (9,12,24)  (9,13,23)  (9,14,22)  (9,15,21)  (9,16,20)  (9,17,19)  (9,18,18)
(10,11,24) (10,12,23) (10,13,22) (10,14,21) (10,15,20) (10,16,19) (10,17,18)
(11,11,23) (11,12,22) (11,13,21) (11,14,20) (11,15,19) (11,16,18) (11,17,17)
(12,12,21) (12,13,20) (12,14,19) (12,15,18) (12,16,17)
(13,13,19) (13,14,18) (13,15,17) (13,16,16)
(14,14,17) (14,15,16)
(15,15,15)
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