Supplementary Problems for

Combinatorics Problems and Solutions Second Edition

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Supplementary Combinatorics Problems

Problem 1. Show that the number of ways in which two people can divide $2n$ things of one kind, and $2n$ of another kind, and $2n$ of a third kind, so that each person gets $3n$ things is $3n^2 + 3n + 1$.

Answer. All 7 ways for $n = 1$ with A, B, C being the 3 kinds of things, is shown in table 1. The table

	person 1 ABC	person 2 ABC
1	111	111
2	021	201
3	012	210
4	201	021
5	102	120
6	210	012
7	120	102

Table 1: All 7 ways for $n = 1$ in problem 1.

provides the perspective to state the problem a different way: In how many ways can you distribute $3n$ identical balls into 3 distinct bins so that no bin contains more than $2n$ balls?

By way #2 of the twelve fold way, the number of ways to distribute $3n$ identical balls into 3 distinct bins is $\binom{3n+2}{2}$.

The number of ways to distribute the balls so that at least one bin has more than $2n$ balls is $3\binom{n+1}{2}$. To get this expression set aside $2n+1$ of the $3n$ balls and distribute the remaining $n-1$ balls into the 3 bins in $\binom{n+1}{2}$ ways. The remaining $2n+1$ balls can then be put into one of the bins in 3 ways.

The number of ways to distribute the balls so that no bin has more than $2n$ balls is then

$$\binom{3n+2}{2} - 3\binom{n+1}{2} = 3n^2 + 3n + 1$$

Problem 2. Give a combinatorial proof of the following identity

$$\sum_{k=0}^{n} \binom{n-k+m-1}{m-1} = \binom{n+m}{m}$$

Answer. $\binom{n+m}{m}$ is equal to the number of ways to distribute n indistinguishable objects into $m+1$ distinguishable bins. Pick one of the bins. Over the set of all distributions that bin will hold between 0 and n objects. If it holds k objects then the remaining $n-k$ objects will be distributed into the other m bins in $\binom{n-k+m-1}{m-1}$ ways. Summing this over all the values of k we get the identity.

Problem 3. Given a collection of n identical red balls, n identical green balls and n identical blue balls, in how many ways can the $3n$ balls be distributed into 3 bins such that each bin contains exactly n balls?

Answer. We only need to count the number of ways to distribute n balls into each of two bins since the remaining n balls will then go into the remaining bin. This is equivalent to asking for the number of ways to distribute n identical balls into bins labeled R_1, G_1, B_1 and n identical balls into bins labeled R_2, G_2, B_2 such that $|R_1| + |R_2| \leq n$, $|G_1| + |G_2| \leq n$, $|B_1| + |B_2| \leq n$. The vertical bars around a bin label means the number of balls in that bin. Without these restrictions each of the distributions of n balls can be done in $\binom{n+2}{2}$ ways so the total number of ways to distribute the $2n$ balls without restrictions is $\binom{n+2}{2}\binom{n+2}{2}$. From this we have to subtract the number of distributions in which one of the conditions $|R_1|+|R_2| > n$, $|G_1| + |G_2| > n$, $|B_1| + |B_2| > n$ holds. Note that since there are $2n$ balls only one of the conditions can hold for a given distribution. Suppose for example that we have $|R_1| + |R_2| = n + k$ where $k = 1, , 2, \ldots, n$. The rest of the $n - k$ balls can be distributed into the remaining 4 bins in $\binom{n-k+3}{3}$ ways. Summing this over all values of k we get the total number of ways that we can have $|R_1| + |R_2| > n$. Using the combinatorial identity proven in the previous problem we have

$$\sum_{k=1}^{n} \binom{n-k+3}{3} = \binom{n+3}{4}$$

To put this in the form of the identity change the summation index to $k' = k - 1$. So that we have (dropping the prime on k)

$$\sum_{k=0}^{n-1} \binom{n-1-k+3}{3} = \binom{n-1+4}{4} = \binom{n+3}{4}$$

The total number of ways to distribute $3n$ balls into 3 bins such that each bin contains exactly n balls is then

$$a(n) = \binom{n+2}{2}\binom{n+2}{2} - 3\binom{n+3}{4}$$

where the 3 multiplying $\binom{n+3}{4}$ comes from the fact that we 3 conditions that have to be accounted for. We can also write the answer without the binomials as

$$a(n) = \frac{1}{8}(n+1)(n+2)(n^2+3n+4)$$

Table 2 shows the value of $a(n)$ for $n = 0, 1, \ldots, 8$ and table 3 shows the 21 ways to distribute the balls $\{rrggbb\}$ into 3 bins with 2 balls per bin.

n	0	1	2	3	4	5	6	7	8
a(n)	1	6	21	55	120	231	406	666	1035

Table 2: $a(n)$ values.

Bin 1	Bin 2	Bin 3
rr	gg	bb
rr	gb	gb
rr	bb	gg
rg	rg	bb
rg	rb	gb
rg	gb	rb
rg	bb	rg
rb	rg	gb
rb	rb	gg
rb	gg	rb
rb	gb	rg
gg	rr	bb
gg	rb	rb
gg	bb	rr
gb	rr	gb
gb	rg	rb
gb	rb	rg
gb	gb	rr
bb	rr	gg
bb	rg	rg
bb	gg	rr

Table 3: The 21 ways to distribute the balls {rrggbb} into 3 bins with 2 balls per bin.

Problem 4. In how many ways can 6 lilies, 7 roses and 10 tulips be arranged in a row so that each lily is between a rose and a tulip, and there are no roses and tulips next to each other?

Answer. Let L represent a lily, and R and T represent a group of roses and tulips respectively. There are then 2 possible arrangements: $RLTLRLTLRLTLR$, $TLRLTLRLTLRLT$. Call these arrangements 1 and 2 respectively. In arrangement 1 there are 4 R groups and 3 T groups. The ways to divide 7 roses into 4 groups with at least one in each group is $\binom{6}{3}$. The ways to divide 10 tulips into 3 groups with at least one in each group is $\binom{9}{2}$. The total number of ways to create arrangement 1 is then $\binom{6}{3}\binom{9}{2}$. In arrangement 2 there are 3 R groups and 4 T groups. The ways to divide 7 roses into 3 groups with at least one in each group are $\binom{6}{2}$. The number of ways to divide 10 tulips into 4 groups with at least one in each group is $\binom{9}{3}$. The total number of ways to create arrangement 3 is $\binom{6}{2}\binom{9}{3}$. The total number of arrangements is then

$$\binom{6}{3}\binom{9}{2} + \binom{6}{2}\binom{9}{3} = 1980$$

In quantum mechanics, particles (electrons, protons, atoms, etc.) that are bound by a potential energy function will have discrete energy levels. A particle in a box where the walls are infinite potential energy barriers is probably the simplest example. In the one dimensional case the particle is confined to a one dimensional region of some fixed length. The energy levels of the particle are limited to the values $E_n = an^2$ where a is a constant and $n = 1, 2, 3 \ldots$.

Another example is the energy levels of an electron in a hydrogen atom. The electron is limited to the energies $E_n = -13.6/n^2$ where $n = 1, 2, 3 \ldots$. A system where the energy levels are equally spaced is the quantum harmonic oscillator which corresponds to the classical system of a particle oscillating on the end of a spring. Here the energy levels have the form $E_n = an + b$ where a and b are constants and $n = 0, 1, 2, \ldots$.

So in general a particle in a quantum system will have one of a set of discrete energy levels E_n. At each energy level there will be a finite set of states the particle can be in. These states may correspond for example to allowed angular momenta for an electron bound to an atom. For the sake of the following discussion, you can picture the particles as being organized into boxes on a set of shelves. Each shelf is an energy level and the

the boxes on the shelf are the states. In general each shelf may have a different number of boxes.

The properties of a system composed of a very large number particles is determined by the way the particles distribute themselves among the energy levels and states. That distribution is determined by the total energy of the system and by the type of particles. In quantum mechanics there are two types of particles called bosons and fermions with very different rules for how they may occupy states.

Bosons have no restrictions on how many of them may occupy the same state simultaneously. Any number of them may bunch up together in the same state. Fermions on the other hand are more standoffish. Only one fermion may occupy a given state at a time. These properties of bosons and fermions determine the total number of ways they can be distributed among the states.