Coin Toss Runs

Stefan Hollos and J. Richard Hollos

This document lists formulas for calculating run probabilities. For background on these formulas see The Coin Toss: The Hydrogen Atom of Probability by Stefan Hollos and J. Richard Hollos.

First Run of r Consecutive Heads

$p =$ probability of heads

$q = 1 - p =$ probability of tails

$f_n =$ probability the run occurs for the first time on toss n

$f_{r-1} = 0$

$f_r = p^r$

$f_n = qp^r$ for $r < n \leq 2r$

$f_n = q \sum_{k=1}^{r} p^{k-1} f_{n-k}$ for $n > 2r$

$F(z) =$ generating function for f_n

$F(z) = \sum_{n=0}^{\infty} f_n z^n$

$F(z) = \frac{p^r z^r (1-pz)}{1-z+qp^r z^{r+1}}$

$\mu =$ mean number of tosses required for a run

$\mu = F'(1) = \frac{1-p^r}{qp^r}$

$\sigma^2 =$ variance of number of tosses required for a run

$\sigma^2 = \frac{1}{(qp^r)^2} - \frac{2r+1}{qp^r} - \frac{p}{q^2}$

$g_n =$ conjugate probability (not a true probability)

$g_n = \frac{f_n}{p^n}$

$g_{r-1} = 0$

$g_r = 1$

$g_n = \alpha(\alpha+1)^{n-r-1}$ for $r < n \leq 2r$

$g_n = \alpha \sum_{k=1}^{r} g_{n-k}$ for $n > 2r$

$\alpha = \frac{q}{p} = \frac{1}{p} - 1$

$G(z) =$ generating function for g_n

$F(z) = \sum_{n=0}^{\infty} g_n z^n$

$G(z) = \frac{z^r(1-z)}{1-(\alpha+1)z+\alpha z^{r+1}}$

$G(pz) = F(z)$

Example: $r = 1$

$G(z) = \frac{z}{1-\alpha z}$

$g_n = \alpha^{n-1}$ for $n > 0$

$f_n = p^n g_n = pq^{n-1} =$ geometric distribution

Example: $r = 2$

$G(z) = \frac{z^2}{1-\alpha z-\alpha z^2}$

$g_2 = 1$

$g_3 = \alpha$

$g_4 = \alpha(\alpha+1)$

$g_5 = \alpha^2(\alpha+2)$

$g_6 = \alpha^2(\alpha^2+3\alpha+1)$

$g_7 = \alpha^3(\alpha+1)(\alpha+3)$

$g_8 = \alpha^3(\alpha^3+5\alpha^2+6\alpha+1)$

$g_n = \alpha g_{n-1} + \alpha g_{n-2}$

$g_n = \frac{z_1^{n-1}-z_2^{n-1}}{z_1-z_2}$

$z_1 = \frac{1}{2}(\alpha + \sqrt{\alpha^2+4\alpha})$

$z_2 = \frac{1}{2}(\alpha - \sqrt{\alpha^2+4\alpha})$

$f_n = p^n g_n$

Example: $r = 3$

$G(z) = \frac{z^3}{1-\alpha z-\alpha z^2-\alpha z^3}$

$g_3 = 1$

$g_4 = \alpha$

$g_5 = \alpha(\alpha + 1)$

$g_6 = \alpha(\alpha + 1)^2$

$g_7 = \alpha^2(\alpha^2 + 3\alpha + 3)$

$g_8 = \alpha^2(\alpha^3 + 4\alpha^2 + 6\alpha + 2)$

$g_n = \alpha g_{n-1} + \alpha g_{n-2} + \alpha g_{n-3}$

$f_n = p^n g_n$

Multiple Runs of r Consecutive Heads

$f_n^{(k)}$ = probability the run occurs for the k^{th} time on toss n

$f_n^{(2)} = \sum_{i=1}^{n-1} f_i f_{n-i}$

$f_n^{(k)} = \sum_{i=1}^{n-1} f_i f_{n-i}^{(k-1)}$

$F^{(k)}(z)$ = generating function for $f_n^{(k)}$

$F^{(k)}(z) = \sum_{n=0}^{\infty} f_n^{(k)} z^n$

$F^{(k)}(z) = F^k(z)$ see previous section for $F(z)$

$\mu^{(k)}$ = mean number of tosses required for k runs

$\mu^{(k)} = k\mu$ see previous section for μ

$(\sigma^{(k)})^2$ = variance of number of tosses required for k runs

$(\sigma^{(k)})^2 = k\sigma^2$ see previous section for σ^2

$g_n^{(k)} = \frac{f_n^{(k)}}{p^n}$

$G^{(k)}(z)$ = generating function for $g_n^{(k)}$

$G^{(k)}(z) = \sum_{n=0}^{\infty} g_n^{(k)} z^n$

$G^{(k)}(z) = G^k(z)$ see previous section for $G(z)$

$G^{(k)}(pz) = F^{(k)}(z)$

Example: $r = 1$

$G^{(k)}(z) = \frac{z^k}{(1-\alpha z)^k}$

$g_n^{(k)} = \binom{n-1}{k-1}\alpha^{n-k}$

$f_n^{(k)} = p^n g_n^{(k)} = \binom{n-1}{k-1}q^{n-k}p^k =$ negative binomial distribution

Example: $r = 2$

$G^{(k)}(z) = \frac{z^{2k}}{(1-\alpha z-\alpha z^2)^k}$

$g_{2k-1}^{(k)} = 0$

$g_{2k}^{(k)} = 1$

$g_{2k+1}^{(k)} = k\alpha$

$(n-2k)g_n^{(k)} = (n-k-1)\alpha g_{n-1}^{(k)} + (n-2)\alpha g_{n-2}^{(k)}$

$f_n^{(k)} = p^n g_n^{(k)}$

$f_{2k-1}^{(k)} = 0$

$f_{2k}^{(k)} = p^{2k}$

$f_{2k+1}^{(k)} = kqp^{2k}$

$(n-2k)f_n^{(k)} = (n-k-1)qf_{n-1}^{(k)} + (n-2)qpf_{n-2}^{(k)}$

For $k = 2$:

$g_3^{(2)} = 0$

$g_4^{(2)} = 1$

$g_5^{(2)} = 2\alpha$

$g_6^{(2)} = \alpha(3\alpha + 2)$

$g_7^{(2)} = \alpha^2(4\alpha + 6)$

$g_8^{(2)} = \alpha^2(5\alpha^2 + 12\alpha + 3)$

For $k = 3$:

$g_5^{(3)} = 0$

$g_6^{(3)} = 1$

$g_7^{(3)} = 3\alpha$

$g_8^{(3)} = 3\alpha(2\alpha + 1)$

$g_9^{(3)} = 2\alpha^2(5\alpha + 6)$

$g_{10}^{(3)} = 3\alpha^2(5\alpha^2 + 10\alpha + 2)$

For $k = 4$:

$g_7^{(4)} = 0$

$g_8^{(4)} = 1$

$g_9^{(4)} = 4\alpha$

$g_{10}^{(4)} = 2\alpha(5\alpha + 2)$

$g_{11}^{(4)} = 20\alpha^2(\alpha + 1)$

$g_{12}^{(4)} = 5\alpha^2(7\alpha^2 + 12\alpha + 2)$

Run Probability for r Consecutive Heads

$a_n =$ probability of a run on the n^{th} toss

$a_n + pa_{n-1} + p^2 a_{n-2} + \cdots + p^{r-1} a_{n-r+1} = p^r$

$a_n = f_n a_0 + f_{n-1} a_1 + f_{n-2} a_2 + \cdots + f_1 a_{n-1}$

$a_n = f_n + f_n^{(2)} + f_n^{(3)} + \cdots$

$A(z) =$ generating function for a_n

$A(z) = \sum_{n=0}^{\infty} a_n z^n$

$A(z) = \frac{1}{1-F(z)}$

$A(z) - 1 = \frac{p^r z^r}{(1-z)(1+pz+p^2z^2+\cdots+p^{r-1}z^{r-1})}$

$A(z) - 1 = \frac{p^r z^r (1-pz)}{(1-z)(1-p^r z^r)}$

$A(z) = \frac{1-z+qp^r z^{r+1}}{(1-z)(1-p^r z^r)}$

Let $k = n \bmod r$ and $\mu = \frac{1-p^r}{qp^r}$ then

$\mu a_n = 1 + p^{n-r+1}(1-p^{r-1})/(1-p) \qquad k = 0$

$\mu a_n = 1 - p^{n-k} \qquad k \neq 0$

Example: $r = 1$

$A(z) = \frac{1-qz}{1-z}$

$A(z) = 1 + pz + pz^2 + \cdots$

$a_n = p \qquad n > 0$

Example: $r = 2$

$A(z) = \frac{1-qz-qpz^2}{(1-z)(1+pz)}$

$a_1 = 0$

$a_2 = p^2$

$a_3 = qp^2$

$a_4 = \frac{p^2(1+p^3)}{1+p}$

$a_5 = \frac{p^2(1-p^4)}{1+p}$

$a_n = \frac{p^2(1+(-1)^n p^{n-1})}{1-p}$

Head or Tail Run Probabilities

$a_n =$ probability of a head run of length h on the n^{th} toss

$A(z) =$ generating function for a_n

$A(z) = \sum_{n=0}^{\infty} a_n z^n$

$A(z) = \frac{1-z+qp^h z^{h+1}}{(1-z)(1-p^h z^h)}$

$b_n =$ probability of a tail run of length t on the n^{th} toss

$B(z) =$ generating function for b_n

$B(z) = \sum_{n=0}^{\infty} b_n z^n$

$B(z) = \frac{1-z+pq^t z^{t+1}}{(1-z)(1-q^t z^t)}$

$c_n =$ probability of a head run of length h or a tail run of length t on the n^{th} toss

$c_n = a_n + b_n$

$C(z) =$ generating function for c_n

$C(z) = \sum_{n=0}^{\infty} c_n z^n$

$C(z) = \frac{1-z+pq^t z^{t+1}}{(1-z)(1-q^t z^t)}$

$C(z) = A(z) + B(z) - 1$

$C(z) = \frac{1-z+p^h q z^{h+1}+pq^t z^{t+1}-p^h q^t z^{h+t}}{(1-z)(1-p^h z^h)(1-q^t z^t)}$

$f_n =$ probability of head run of length h or tail run of length t occuring on the n^{th} toss for the first time

$F(z) =$ generating function for f_n

$F(z) = \sum_{n=0}^{\infty} f_n z^n$

$F(z) = \frac{C(z)-1}{C(z)}$

$F(z) = \frac{p^h z^h(1-pz)(1-q^t z^t)+q^t z^t(1-qz)(1-p^h z^h)}{1-z+p^h q z^{h+1}+pq^t z^{t+1}-p^h q^t z^{h+t}}$

$\mu =$ mean number of tosses required for head run of length h or tail run of length t

$\mu = F'(1)$

$\mu = \frac{(1-p^h)(1-q^t)}{p^h q+pq^t-p^h q^t}$

$\sigma^2 =$ variance of number of tosses required for head run of length h or tail run of length t

$\sigma^2 = \mu(1-\mu) + F''(1)$

Run Examples

All ways of getting 2 heads for the first time on the 8^{th} toss (there are 13):

10101011
10100011
10010011
10001011
10000011
01010011
01001011
01000011
00101011
00100011
00010011
00001011
00000011

All ways of getting 3 heads for the first time on the 10^{th} toss (there are 44):

1101100111
1101010111
1101000111
1100110111
1100100111
1100010111
1100000111
1011010111
1011000111
1010110111
1010100111
1010010111
1010000111
1001100111
1001010111
1001000111
1000110111
1000100111
1000010111
1000000111
0110110111
0110100111
0110010111
0110000111
0101100111
0101010111
0101000111
0100110111
0100100111
0100010111
0100000111
0011010111
0011000111
0010110111
0010100111
0010010111
0010000111
0001100111
0001010111
0001000111
0000110111
0000100111
0000010111
0000000111